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Axial dispersion in the Taylor vortex: an approach using the slaving principle

A N Yannacopoulos†§, G Rowlands† and G P King‡
† Physics Department, University of Warwick, Coventry CV4 7AL, UK
‡ Mathematics Institute, Department of Engineering, University of Warwick, Coventry CV4 7AL, UK

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Abstract. An approach to the problem of Taylor dispersion in the Taylor vortex is given where use of the slaving principle of Haken is made. This new approach is also used in the study of the axial dispersion of particles with inertia in the Taylor vortex. An homogenization approach to the derived dispersion equation is given, making use of the multiple scales perturbation theory.

1. Introduction

In 1953 Taylor studied the problem of advection and diffusion of a tracer by a Poisseuille flow in a long pipe. Taylor showed analytically and experimentally that the effect of the Poisseuille flow leads to enhanced dispersion in the axial direction. In more detail, Taylor proved that the tracer experiences a mean flow U, equal to the average of the velocity of the Poisseuille flow over the section of the pipe and that an effective diffusion in the axial direction of the flow is given by

$$D_{\mathrm{eff}} = D + \frac{U^2 a^2}{48D}$$

where D is the molecular diffusivity and a is the diameter of the pipe.

The results stirred the interest of the fluid dynamics community and since 1953 there has been a continued interest in the problem, arising in different situations. Taylor's results have been re-derived using a number of different and more sophisticated techniques and extended in a number of ways such as for instance in pipes with non-constant diameter, biological situations etc. One of the first derivations of the Taylor result was obtained using the method of moments, where by a proper averaging procedure an effective diffusion coefficient was defined from the second moment of the probability distribution of the tracer. More modern approaches employ ideas from dynamical systems theory and in particular centre manifold theory and re-derive Taylor's original result as a centre manifold expansion around the neutral equilibrium of a properly defined dissipative dynamical system. A very concise introduction to the subject is given in the review article by Young and Jones [7].

In this paper, we present an alternative approach to the problem of Taylor dispersion using an application of the slaving principle [1]. Although conceptually close to the centre manifold approach used by Mercer and Roberts [2], our method is simpler to use and

§ Present address: School of Mathematics and Statistics, University of Birmingham, Edgbaston, Birmingham B15 2TT, UK.

extends naturally to the case where the flow field depends on the coordinate along which the dispersion is studied. We also feel that it is more physically oriented than the central manifold approach and conceptually easier to apply to more realistic flows, giving explicit results, and space dependent effective transport coefficients (which is not so straightforward using the centre manifold approach). The paper by Young and Jones [7] offers a detailed discussion of the central manifold approach and mentions its connection with the slaving principle. Both these papers only apply their results to the simple shear flow. Here we use the shear flow as an example to illustrate our method and then go on to apply it to the problem of axial dispersion in the Taylor vortex flow, both for a passive tracer and a tracer with inertia. This leads to an equation of diffusion type with spatially varying transport coefficients. Our results are easily generalizable for general periodic flows. A way of homogenizing these space dependent transport coefficients using multiple scale perturbation theory is given.

2. Taylor dispersion in a simple shear flow

In this section we illustrate the basic idea of our approach with the simple example of a shear flow. Consider the simple shear flow in a long plane channel of width $2b$ so that

$$\boldsymbol{u} = u(y)\boldsymbol{x}. \tag{1}$$

The probability distribution for a tracer advected with this flow under the effect of molecular diffusion satisfies the advection diffusion equation

$$\frac{\partial P}{\partial t} + u(y)\frac{\partial P}{\partial x} = D\frac{\partial^2 P}{\partial x^2} + D\frac{\partial^2 P}{\partial y^2} \tag{2}$$

with D taken as a constant and with no flux boundary conditions at the walls of the channel

$$\frac{\partial P}{\partial y} = 0 \qquad \text{for } y = \pm b. \tag{3}$$

We want to study the behaviour of this probability distribution for large distances along the channel.

Because of the boundary conditions in the y direction the operator $\mathcal{A} = D(\partial^2/\partial y^2)$ is a dissipative operator with discrete spectrum whose eigenfunctions form a complete set in a properly chosen functional space. We can then expand P as

$$P = \sum_m A_m(x,t) w_m(y) \tag{4}$$

where $w_m(y)$ are the eigenfunctions of the operator $\mathcal{A}$. In the particular case examined here we have

$$w_m(y) = \cos\left(\frac{m\pi}{2b}y + \frac{m\pi}{2}\right) \tag{5}$$

and the eigenvalue spectrum is

$$\lambda_m = \frac{m^2\pi^2}{4b^2}. \tag{6}$$

Substituting this expansion for P into the advection diffusion equation and using the orthogonality of the modes we find that

$$\frac{\partial A_n}{\partial t} + \sum_m \frac{\Delta_{nm}}{\sigma_n}\frac{\partial A_m}{\partial x} = -D\lambda_n A_n + D\frac{\partial^2 A_n}{\partial x^2} \tag{7}$$

where

$$\sigma_n = \int_{-b}^{b} w_n^2 \, \mathrm{d}y \tag{8}$$

$$\Delta_{mn} = \int_{-b}^{b} u(y) w_n w_m \, \mathrm{d}y. \tag{9}$$

In the present case $\sigma_n = b$ if $n \neq 0$ and $\sigma_0 = 2b$.

From these equations we notice that except for the A_0 mode, all other modes experience a damping because of the diffusion in the y direction. The damping is seen to be stronger the higher the mode number. The A_0 mode is a neutral mode. As a result of this the y dependent modes will be short lived in comparison with the y independent one. This implies that the long time behaviour of the system will be dominated by the amplitude of the neutral mode which we now consider to be an order parameter. It is then possible to obtain an evolution equation for this order parameter which would in essence be an equation for the evolution of the y averaged probability distribution of the tracer.

The equation for the A_0 mode is

$$\frac{\partial A_0}{\partial t} + \sum_m \frac{\Delta_{0m}}{\sigma_0} \frac{\partial A_m}{\partial x} = D \frac{\partial^2 A_0}{\partial x^2} \tag{10}$$

which contains the interaction of the neutral mode with the other modes. We use the slaving principle approximation on the other modes, which to the first-order simply means neglecting the time derivatives in equation (7) as these modes are heavily damped [1]. We then end up with a set of linear elliptic equations for the amplitudes of the other modes of the form

$$\sum_{m \neq 0} \frac{\Delta_{nm}}{\sigma_n} \frac{\partial A_m}{\partial x} + \frac{\Delta_{n0}}{\sigma_n} \frac{\partial A_0}{\partial x} = -D\lambda_n A_n + D \frac{\partial^2 A_n}{\partial x^2} \tag{11}$$

which have to be solved to give us the relation of the amplitude of the damped modes in terms of the neutral mode. Formally we can solve these equations to give

$$A_n = a_n \frac{\partial A_0}{\partial x} + b_n \frac{\partial^2 A_0}{\partial x^2} + \mathrm{HDT} \tag{12}$$

where HDT denotes higher derivative terms. From the nature of equation (10) we only need the first term to give an effective diffusion equation for A_0. This method gives us the best Fokker–Planck equation for A_0 in that we have neglected higher-order derivatives of A_0. The first term in this expansion gives

$$A_n = -\frac{4b}{n^2 \pi^2 D} \Delta_{0n} \frac{\partial A_0}{\partial x} \tag{13}$$

which is formally equivalent to neglecting the spatial derivatives of the damped modes in the slaving relations. Note that we do not neglect the spatial derivative of the neutral mode which is considered to be an order of magnitude larger that the spatial derivatives of the other modes.

Substituting this result in (10) we obtain an equation for A_0 only, of the form

$$\frac{\partial A_0}{\partial t} + \frac{\Delta_{00}}{2b} \frac{\partial A_0}{\partial x} = \left(D + \sum_{m \neq 0} \frac{\Delta_{m0}^2}{2b} \frac{4b}{m^2 \pi^2 D} \right) \frac{\partial^2 A_0}{\partial x^2} \tag{14}$$

which is an advection diffusion equation of the Fokker–Planck type. The coefficient in front of the drift term is simply the average velocity in the y direction. The effective diffusion coefficient depends on the details of the shear flow. For concreteness let us assume that

$u(y) = 3/2U(1-(y/b)^2)$. In this case $\Delta_{0m} = -(12bU/m^2\pi^2)((-1)^m+1)$ and the effective diffusion coefficient takes the form

$$D_e = D + \frac{36U^2b^2}{2\pi^6 D}\sum_{n=1}^{\infty}\frac{1}{n^6} = D + \frac{2}{105}\frac{U^2b^2}{D} \tag{15}$$

which is exactly the result obtained in [2] using a centre manifold approach. Note that for $D \to 0$ the method breaks down as all the modes then vary on the same time-scale.

3. Axial dispersion in the Taylor vortex using the slaving principle approach

We now apply the formal procedure highlighted above for the simple shear flow, to the case of the Taylor vortex. The probability distribution for a passive tracer in such a flow satisfies the following advection diffusion equations

$$\frac{\partial P}{\partial t} + \frac{1}{r}\frac{\partial(ru_rP)}{\partial r} + \frac{\partial(u_zP)}{\partial z} = D_{rr}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial P}{\partial r}\right) + D_{zz}\frac{\partial^2 P}{\partial z^2} \tag{16}$$

where an averaging over the azimuthal direction has been performed. (We have assumed that the Taylor vortex flow is independent of the azimuthal angle.) We impose no-flux boundary conditions at the cylinder walls, which because of the fact that $u_r = 0$ at the walls, simply become

$$\frac{\partial P}{\partial r} = 0 \qquad r = r_1, r_2 \tag{17}$$

where r_1 and r_2 are the radii of the inner and outer cylinders respectively. We assume that the cylinders are infinitely long.

The diffusion operator with the no-flux boundary conditions in the r direction is a dissipative operator, with a discrete spectrum. The eigenfunctions of this operator form a complete basis for a properly chosen functional space. We then can write

$$P = \sum_m A_m(z,t)w_m(r) \tag{18}$$

where w_n is a solution of the eigenvalue problem

$$D\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}w}{\mathrm{d}r}\right) = -\lambda w \tag{19}$$

with

$$\frac{\mathrm{d}w}{\mathrm{d}r} = 0 \qquad r = r_1, r_2. \tag{20}$$

This eigenvalue problem can easily be solved in terms of Bessel functions. The eigenfunctions are given by $w = J_0(br) + \alpha Y_0(br)$ where $b = (\lambda/D)$ are the solutions of the equation

$$J_1(br_1)Y_0(br_2) - J_1(br_2)Y_0(br_1) = 0. \tag{21}$$

This equation has an infinite number of discrete solutions. From the asymptotic expansions of the Bessel functions it is easy to see that the solutions for large m behave as $b_m \propto m$ and so the eigenvalue spectrum of this operator behaves as $\lambda_m \propto m^2$ similarly to the case of the plane channel. Furthermore, the eigenfunctions corresponding to different eigenvalues are orthogonal.

We now assume that the Taylor vortex flow has a velocity field of the form

$$u_r = R_1(r)R_2(z) \qquad u_z = Z_1(r)Z_2(z) \tag{22}$$

where R_1, R_2, Z_1 and Z_2 are given functions and R_2 and Z_2 are periodic functions of z. Substituting the expansion (18) into (16) and using the orthogonality of the eigenfunctions we obtain

$$\frac{\partial A_n}{\partial t} - \sum_{m=0}^{\infty} \frac{\Lambda_{mn}}{\sigma_n} A_m R_2(z) + \sum_{m=0}^{\infty} \frac{\bar{\Lambda}_{mn}}{\sigma_n} \frac{\partial}{\partial z}(A_m Z_2(z)) = -\lambda_n A_n + D \frac{\partial^2 A_n}{\partial z^2} \tag{23}$$

where

$$\Lambda_{mn} = \int_{r_1}^{r_2} r Z_1(r) w_m \frac{\mathrm{d}w_n}{\mathrm{d}r}\,\mathrm{d}r \qquad \bar{\Lambda}_{mn} = \int_{r_1}^{r_2} r R_1(r) w_m w_n\,\mathrm{d}r$$

and

$$\sigma_n = \int_{r_1}^{r_2} r w_n^2\,\mathrm{d}r.$$

As in the previous case we have a neutral mode A_0 which is r-independent, which serves as an order parameter, and all the r-dependent modes are damped for sufficiently large time as a result of the action of the dissipative diffusion operator. The equation for the evolution of the neutral mode is

$$\frac{\partial A_0}{\partial t} + \sum_{m=0}^{\infty} \frac{\bar{\Lambda}_{m0}}{\sigma_0} \frac{\partial}{\partial z}(A_m Z_2(z)) = D \frac{\partial^2 A_0}{\partial z^2} \tag{24}$$

which contains the interaction of A_0 with the other modes. We now apply the slaving principle to find expressions for the other modes in terms of A_0. In analogy with the example of the shear flow we neglect the time derivatives of the 'fast' modes as well as their spatial derivatives which are considered to be an order of magnitude smaller than their values. This leaves us with a linear system of equations to solve for the instantaneous values of the amplitudes of the 'fast' modes

$$\sum_{m'=1}^{\infty} \left(\frac{\bar{\Lambda}_{m'm}}{\sigma_m} \frac{\mathrm{d}Z_2}{\mathrm{d}z} - \frac{\Lambda_{m'm}}{\sigma_m} R_2 \right) A_{m'} + \lambda_m A_m = -\frac{\bar{\Lambda}_{0m}}{\sigma_m} \frac{\partial}{\partial z}(Z_2 A_0) + \frac{\Lambda_{0m}}{\sigma_m} A_0 R_2. \tag{25}$$

This system is no longer diagonal as it was in the previous example, and therefore solutions have to be found numerically. However, we can exploit the following considerations to obtain an approximation to the exact slaving relations. We mentioned earlier that the eigenvalue spectrum $\lambda_m \sim m^2$. On the other hand an elementary calculation shows that all the coefficients Λ_{mn} and $\bar{\Lambda}_{mn}$ are of order one and in particular that

$$\left| \frac{\bar{\Lambda}_{mm}}{\sigma_m} \right| \leqslant C_1 = \sup_{r \in (r_1, r_2)} |Z_1(r)|$$

$$\left| \frac{\Lambda_{mm}}{\sigma_m} \right| \leqslant C_2 = \frac{1}{2} \sup_{r \in (r_1, r_2)} \left| \frac{1}{r} \frac{\mathrm{d}}{\mathrm{d}r}(r R_1(r)) \right|.$$

We can see then that in system (25) the diagonal term is going to be dominant and all the off-diagonal terms will introduce small corrections. It is reasonable then to propose the following iterative scheme for the solution of (25)

$$A_m^{(n+1)} = G(m, z) H(z, m, A_0) + G(m, z) \sum_{m \neq m', 0} F(m, m', z) A_{m'}^{(n)} \tag{26}$$

where

$$H(z, m, A_0) = -\frac{\bar{\Lambda}_{0m}}{\sigma_m} \frac{\partial Z_2 A_0}{\partial z} + \frac{\Lambda_{0m}}{\sigma_m} A_0 R_2$$

$$G(m, z) = \left(\lambda_m + \frac{\bar{\Lambda}_{mm}}{\sigma_m}\frac{\mathrm{d}Z_2}{\mathrm{d}z} - \frac{\Lambda_{mm}}{\sigma_m}R_2\right)^{-1}$$

$$F(m, m', z) = \frac{\bar{\Lambda}_{m'm}}{\sigma_m}\frac{\mathrm{d}Z_2}{\mathrm{d}z} - \frac{\bar{\Lambda}_{m'm}}{\sigma_m}R_2.$$

In this paper we only consider the zeroth approximation to this iterative scheme which simply gives

$$A_m = \left(\lambda_m + \frac{\bar{\Lambda}_{mm}}{\sigma_m}\frac{\mathrm{d}Z_2}{\mathrm{d}z} - \frac{\bar{\Lambda}_{mm}}{\sigma_m}R_2\right)^{-1}\left(-\frac{\bar{\Lambda}_{0m}}{\sigma_m}\frac{\partial Z_2 A_0}{\partial z} + \frac{\Lambda_{0m}}{\sigma_m}A_0 R_2\right). \tag{27}$$

The results obtained using the next iteration are presented in the appendix of the paper. Substituting this into the equation for the evolution of the A_0 mode we find that

$$\frac{\partial A_0}{\partial t} + \frac{\bar{\Lambda}_{00}}{\sigma_0}\frac{\partial}{\partial z}(Z_2(z)A_0) + \sum_{m=1}^{\infty}\frac{\bar{\Lambda}_{m0}\Lambda_{0m}}{\sigma_0\sigma_m}\frac{\partial}{\partial z}(G(m, z)R_2(z)Z_2(z)A_0)$$
$$= D\frac{\partial^2 A_0}{\partial z^2} + \sum_{m=1}^{\infty}\frac{\bar{\Lambda}_{m0}^2}{\sigma_0\sigma_m}\frac{\partial}{\partial z}\left(G(m, z)Z_2(z)\frac{\partial}{\partial z}(Z_2(z)A_0)\right) \tag{28}$$

which we rewrite in the usual form

$$\frac{\partial A_0}{\partial t} + \frac{\partial}{\partial z}(V_e(z)A_0) = \frac{\partial}{\partial z}\left(D_e(z)\frac{\partial A_0}{\partial z}\right) \tag{29}$$

where

$$V_e(z) = \frac{\bar{\Lambda}_{00}}{\sigma_0}Z_2(z) + \left(\sum_{m=1}^{\infty}\frac{\bar{\Lambda}_{m0}\Lambda_{0m}}{\sigma_0\sigma_m}G(m, z)R_2(z) - \sum_{m=1}^{\infty}\frac{\bar{\Lambda}_{m0}^2}{\sigma_0\sigma_m}\frac{\mathrm{d}Z_2(z)}{\mathrm{d}z}\right)Z_2(z)$$

$$D_e(z) = D + \left(\sum_{m=1}^{\infty}\frac{\bar{\Lambda}_{0m}^2}{\sigma_0\sigma_m}G(m, z)\right)Z_2(z)^2$$

which are clearly periodic functions of z. We can further exploit the fact that the λ_m are considerably larger than Λ_{mm} and $\bar{\Lambda}_{mm}$ and expand $G(m, z)$ in a Taylor series. This gives an approximate D_e of the form

$$D_e = D + \left(D_1 + D_2\frac{\mathrm{d}Z_2(z)}{\mathrm{d}z} + D_3 R_2(z)\right)Z_2(z)^2$$

$$D_1 = \sum_{m=1}^{\infty}\frac{\bar{\Lambda}_{0m}^2}{\sigma_0\sigma_m}\frac{1}{\lambda_m}$$

$$D_2 = \sum_{m=1}^{\infty}\frac{\bar{\Lambda}_{0m}^2}{\sigma_0\sigma_m^2}\frac{\bar{\Lambda}_{mm}}{\lambda_m^2}$$

$$D_3 = -\sum_{m=1}^{\infty}\frac{\bar{\Lambda}_{0m}^2}{\sigma_0\sigma_m^2}\frac{\Lambda_{mm}}{\lambda_m^2}$$

and a similar expansion for V_e. The sums involved in the definition of D_1, D_2 and D_3 are clearly seen to converge since the eigenvalue spectrum behaves like m^2. The next iteration will clearly give corrections to this result, which again will be in the form of periodic functions.

Concluding this section, we have applied the slaving principle approach to the problem of diffusion and advection of passive tracer of the same density as the fluid in the Taylor vortex flow and obtained an equation for the evolution of the neutral mode which corresponds to an effective Fokker–Planck equation for the average concentration over the radial direction. Importantly, the transport coefficients are space dependent, thus exhibiting in detail the effect of the Taylor vortex flow on the axial dispersion of the tracer. Explicit forms of these coefficients are given.

4. Effect of particle inertia

In this section we are going to study the effect of particle inertia in the axial dispersion in the Taylor vortex flow. We use a simple model for the inertial effects due to Druzhinin and Ostrovsky [3] according to which the velocity of a small spherical particle v, with small inertia and for small enough times is given by

$$v = u(r,t) + \gamma \frac{Du}{Dt} \tag{30}$$

where the factor γ which is a measure of the inertial effects, is given by

$$\gamma = \frac{2a^2}{9\nu} \frac{\rho_f - \rho_p}{\rho_f} \tag{31}$$

where ν is the viscosity, ρ_f is the fluid density, ρ_p is the density of the particles, a is the radius of the particle, D/Dt is the convective derivative and u is the velocity of the ambient fluid.

For the case of the Taylor vortex flow where the ambient velocity field is

$$u_r = R_1(r)R_2(z) \qquad u_\theta = u_c(r) + T_1(r)T_2(z) \qquad u_z = Z_1(r)Z_2(z)$$

the velocity field that an inertial particle experiences is

$$\begin{aligned} u_{pr} &= R_1(r)R_2(z) + \gamma\left(R_1 Z_1 \frac{\mathrm{d}R_2}{\mathrm{d}z} Z_2 + R_1 \frac{\mathrm{d}R_1}{\mathrm{d}r} R_2^2 - \frac{(u_c + T_1T_2)^2}{r}\right) \\ u_{pz} &= Z_1Z_2 + \gamma\left(R_1 \frac{\mathrm{d}Z_1}{\mathrm{d}r} R_2Z_2 + Z_1^2 Z_2 \frac{\mathrm{d}Z_2}{\mathrm{d}z}\right). \end{aligned}$$

We do not give explicitly the $u_{p\theta}$ component since it will drop out of the advection diffusion equation for the tracer after averaging over the azimuthal direction.

We now highlight the derivation of the equation for axial dispersion for the velocity field experienced by the inertial particles and give expressions for the effective dispersion coefficients these tracers should feel. We see that in general the velocity field the tracer experiences will be of the form

$$u_r = \sum_q R_{1q}(r)R_{2q}(z) \tag{32}$$

$$u_z = \sum_q Z_{1q}(r)Z_{2q}(z). \tag{33}$$

Using this velocity field in the advection diffusion equation we find that the equations for the individual modes are

$$\frac{\partial A_n}{\partial t} + \sum_{mq} \frac{\Lambda^{(s)}_{mnq} - \Lambda_{mnq}}{\sigma_n} R_{2q}(z)A_m + \frac{\partial}{\partial z}\left(\sum_{mq} \frac{\bar{\Lambda}_{mnq}}{\sigma_n} Z_{2q}(z)A_m\right) = -\lambda_n A_n + D\frac{\partial^2 A_n}{\partial z^2} \tag{34}$$

where

$$\Lambda^{(s)}_{mnq} = r_2 w_m(r_2) w_n(r_2) R_{1q}(r_2) - r_1 w_m(r_1) w_n(r_1) R_{1q}(r_1) \tag{35}$$

$$\Lambda_{mnq} = \int_{r_1}^{r_2} r R_{1q} w_m \frac{\mathrm{d}w_n}{\mathrm{d}r}\,\mathrm{d}r \tag{36}$$

$$\bar{\Lambda}_{mnq} = \int_{r_1}^{r_2} r Z_{1q} w_m w_n\,\mathrm{d}r. \tag{37}$$

Note that in this case we have to introduce the terms $\Lambda^{(s)}_{mni}$ as in some cases the velocity field experienced by the inertial particles might not necessarily vanish at the cylinder walls. The use of the slaving principle is done in complete analogy with the case studied in the previous section. Here we only quote the result of the diagonal approximation. This gives

$$A_m = -G(m,z)\left(\frac{\partial}{\partial z}\left(\sum_q \frac{\bar{\Lambda}_{0mq}}{\sigma_m} Z_{2q} A_0\right) + \sum_q \frac{(\Lambda^{(s)}_{0mq} - \Lambda_{0mq})}{\sigma_m} R_{2q} A_0\right) \tag{38}$$

where

$$G(m,z) = \left(\lambda_m + \sum_q \frac{\bar{\Lambda}_{mmq}}{\sigma_m}\frac{\mathrm{d}Z_{2q}}{\mathrm{d}z} + \frac{\Lambda^{(s)}_{mmq} - \Lambda_{mmq}}{\sigma_m} R_{2q}\right)^{-1}. \tag{39}$$

The equation for the A_0 mode is

$$\frac{\partial A_0}{\partial t} + \sum_{mq} \frac{\Lambda^{(s)}_{m0q}}{\sigma_0} R_{2q}(z) A_m + \frac{\partial}{\partial z}\sum_{mq} \frac{\bar{\Lambda}_{m0q}}{\sigma_0} Z_{2q}(z) A_m = D\frac{\partial^2 A_0}{\partial z^2} \tag{40}$$

which after substitution of the expressions for A_m provided by the slaving principle, and some algebraic manipulations can be brought into the standard form

$$\frac{\partial A_0}{\partial t} + F_e(z) A_0 + \frac{\partial}{\partial z}(V_e(z) A_0) = \frac{\partial}{\partial z}\left(D_e(z)\frac{\partial A_0}{\partial z}\right) \tag{41}$$

where

$$F_e(z) = \sum_q \frac{\Lambda^{(s)}_{00q}}{\sigma_0} R_{2q} - \sum_{m\neq 0,q} \frac{\Lambda_{m0q}}{\sigma_0}\frac{\mathrm{d}}{\mathrm{d}z}(R_{2q} G(m,z)) \sum_j \frac{\bar{\Lambda}_{0mj}}{\sigma_m} Z_{2j}$$
$$- \sum_{m,q,j} \frac{\Lambda^{(s)}_{m0q}}{\sigma_0} R_{2q}(z) G(m,z) \frac{\Lambda_{0mj} - \Lambda_{0mj}}{\sigma_m} R_{2qj}$$

$$V_e(z) = \sum_q \frac{\bar{\Lambda}_{00q}}{\sigma_0} Z_{2q} + \sum_{m\neq 0,q,j} \frac{\Lambda^{(s)}_{m0q}}{\sigma_0}\frac{\bar{\Lambda}_{0mj}}{\sigma_m} R_{2q} Z_{2j} G(m,z)$$
$$- \sum_{m\neq 0,q,j} \frac{\bar{\Lambda}_{m0q}}{\sigma_0}\frac{\Lambda^{(s)}_{m0j} - \Lambda_{0mj}}{\sigma_m} Z_{2q} R_{2j} G(m,z)$$
$$- \sum_{m\neq 0,q,j} \frac{\bar{\Lambda}_{m0q}}{\sigma_0}\frac{\bar{\Lambda}_{m0j}}{\sigma_m} Z_{2q}\frac{\mathrm{d}Z_{2j}}{\mathrm{d}z} G(m,z)$$

$$D_e(z) = D + \sum_{m\neq 0,q,j} \frac{\bar{\Lambda}_{m0q}}{\sigma_0}\frac{\bar{\Lambda}_{m0j}}{\sigma_m} G(m,z) Z_{2q} Z_{2j}.$$

In the particular case of the Taylor vortex

$$R_{10} = R_1 \qquad R_{20} = R_2 \qquad R_{11} = \gamma R_1 Z_1 \qquad R_{21} = \frac{\mathrm{d}R_2}{\mathrm{d}z} Z_2$$

$$R_{12} = \gamma R_1 \frac{\mathrm{d}R_1}{\mathrm{d}r} \qquad R_{22} = R_2^2$$

$$R_{13} = -\gamma \frac{u_c^2}{r} \qquad R_{23} = 1 \qquad R_{14} = -2\gamma \frac{u_c T_1}{r} \qquad R_{24} = T_2$$

$$R_{15} = -\gamma \frac{T_1^2}{r} \qquad R_{25} = T_2^2$$

$$Z_{10} = Z_1 \qquad Z_{20} = Z_2 \qquad Z_{11} = \gamma R_1 \frac{\mathrm{d}Z_1}{\mathrm{d}r} \qquad Z_{21} = R_2 Z_2$$

$$Z_{12} = \gamma Z_1^2 \qquad Z_{22} = Z_2 \frac{\mathrm{d}Z_2}{\mathrm{d}z}$$

and

$$\bar{\Lambda}^{(s)}_{mni} = 0 \qquad i = 0, 1, 2, 4, 5$$

$$\bar{\Lambda}^{(s)}_{mn3} = \frac{u_c^2(r_2)}{r_2} - \frac{u_c^2(r_1)}{r_1}.$$

The results in this section show that the radially averaged concentration of tracer, with different density than that of the ambient fluid, in the Taylor vortex evolves following a Fokker–Planck equation of the form (4). It is interesting to note the emergence of the $F_e(z)A_0$ term which arises as a purely inertial effect and gives rise to the possible existence of concentration zones in the flow due to the density difference of the tracer and the fluid. Expressions for the transport coefficients are given which show explicitly the effect of particle inertia.

5. Average diffusion coefficients

We see that in general the effective transport equation for the r-averaged probability distribution has transport coefficients which are periodic functions of z. This reflects the fact that the tracer is advected and diffusing through a periodic array of Taylor vortices (through in effect a periodic medium). In practice an initial concentration which is extended in space will not respond to the fine scales of the periodic transport coefficients. Furthermore, even for initial concentrations resembling delta functions the periodic structure of the transport coefficients will be relevant only for the small time regime when the tracer is within a single Taylor vortex. For more realistic times the particle will experience the effect of many vortices whose effect we model by introducing suitably averaged transport coefficients. In applications, the main interest is, in general, in this time regime, that is the large time (and consequently the large spatial) scales regime. In this section we propose a multiple scales perturbative approach, in the spirit of [4]† which will give us the long time evolution of the probability distribution.

As our results will be valid for any general periodic flow field we start with a general advection diffusion equation of the form

$$\frac{\partial P}{\partial t} + \frac{\partial}{\partial z}\left(V\left(z, \frac{z}{\epsilon}\right) P\right) + A\left(z, \frac{z}{\epsilon}\right) P = \frac{\partial}{\partial z} D\left(z, \frac{z}{\epsilon}\right) \frac{\partial P}{\partial z} \qquad (42)$$

† Where no drift was taken into consideration.

where V is the drift velocity and D is the diffusion coefficient. We define $z_1 = \epsilon^{-1} z_0$ (a fast scale), $z_0 = z$ (a slow scale) and $t_0 = t$, $t_1 = \epsilon^{-1} t_0$ and $t_2 = \epsilon^{-2} t_0$. In this notation z_1 denotes the spatial scale of each single Taylor vortex, whereas z_0 is the scale of a large number of vortices. The transport coefficients are periodic in the scale z_1 and, in our cases discussed earlier, are independent of the z_0 scale but we leave this dependence in, for the sake of generality.

Expanding in these multiple scales and using the ansatz $P = P_0 + \epsilon P_1 + \cdots$ for the probability distribution we obtain to the first few orders

$$\frac{\partial}{\partial z_1} D \frac{\partial P_0}{\partial z_1} = \frac{\partial P_0}{\partial t_2} \tag{43}$$

$$\frac{\partial}{\partial z_0} D \frac{\partial P_0}{\partial z_1} + \frac{\partial}{\partial z_1} D \frac{\partial P_0}{\partial z_0} + \frac{\partial}{\partial z_1} D \frac{\partial P_1}{\partial z_1} - \frac{\partial}{\partial z_1}(V P_0) = \frac{\partial P_1}{\partial t_2} + \frac{\partial P_0}{\partial t_1} \tag{44}$$

$$\frac{\partial}{\partial z_0} D \frac{\partial P_0}{\partial z_0} + \frac{\partial}{\partial z_0} D \frac{\partial P_1}{\partial z_1} + \frac{\partial}{\partial z_1} D \frac{\partial P_1}{\partial z_0} + \frac{\partial}{\partial z_1} D \frac{\partial P_2}{\partial z_1} = \frac{\partial P_0}{\partial t_0} + \frac{\partial P_1}{\partial t_1} + \frac{\partial P_2}{\partial t_2} + \frac{\partial}{\partial z_0}(V P_0) + \frac{\partial}{\partial z_1}(V P_1) + A P_0. \tag{45}$$

Averaging the first equation over t_2 and imposing the periodicity condition on z_1 we find that $\bar{P}_0$ is only dependent on z_0, where the overbar denotes averaging over t_2. Then averaging the second equation over t_1 and t_2 we obtain that

$$D\left(\frac{\partial \bar{\bar{P}}_1}{\partial z_1} + \frac{\partial \bar{\bar{P}}_0}{\partial z_0}\right) + V \bar{\bar{P}}_0 = C(z_0) \tag{46}$$

where the double overbars denote averaging over both t_1 and t_2. Then the next order equation gives, after averaging over both t_1 and t_2 and imposing periodicity in z_1, the consistency condition

$$\frac{\partial \bar{\bar{P}}_0}{\partial t_0} = \frac{\partial}{\partial z_0}\left(\langle D\rangle \frac{\partial \bar{\bar{P}}_0}{\partial z_0} + \left\langle D \frac{\partial \bar{\bar{P}}_1}{\partial z_1}\right\rangle + \langle V\rangle \bar{\bar{P}}_0\right) + \langle A\rangle \bar{\bar{P}}_0 \tag{47}$$

where $\langle\,\rangle$ denotes averaging over the variable z_1 (over a periodic cell). This equation is equivalent to

$$\frac{\partial \bar{\bar{P}}_0}{\partial t_0} = \frac{\partial}{\partial z_0} C(z_0) + \langle A\rangle \bar{\bar{P}}_0. \tag{48}$$

From (46) we can see that $\bar{\bar{P}}_1$ can be given by an ansatz of the form

$$\bar{\bar{P}}_1 = g(z_1) \frac{\partial \bar{\bar{P}}_0}{\partial z_0} + f(z_1) \bar{\bar{P}}_0 + \tilde{P}_0(z_0)$$

where f and g are periodic functions. Substituting this ansatz in (46) we obtain

$$\frac{\partial \bar{\bar{P}}_0}{\partial z_0}\left(1 + \frac{\mathrm{d}g}{\mathrm{d}z_1}\right) + \bar{\bar{P}}_0 \left(\frac{\mathrm{d}f}{\mathrm{d}z_1} + \frac{V}{D}\right) = \frac{C(z_0)}{D} \tag{49}$$

which when averaged over z_1 (taking into account the periodicity of f and g in z_1) gives us the consistency condition that

$$C(z_0) = D_a \frac{\partial \bar{\bar{P}}_0}{\partial z_0} + D_a \left\langle \frac{V}{D} \right\rangle \bar{\bar{P}}_0 \tag{50}$$

and (dropping the double bars) we find that P_0 on the large space and time scale satisfies the equation

$$\frac{\partial P_0}{\partial t_0} = \frac{\partial}{\partial z_0}\left(D_a \frac{\partial P_0}{\partial z_0} + V_a P_0\right) + \langle A \rangle P_0$$
$$D_a = \left\langle \frac{1}{D} \right\rangle^{-1}$$
$$V_a = D_a \left\langle \frac{V}{D} \right\rangle. \tag{51}$$

This is the averaged (homogenized) advection diffusion equation which is valid in the large scales. In the present context, this will be the transport equation for the tracer after it has encountered a large number of rolls. These suitably averaged transport coefficients can easily be calculated from the spatially dependent effective transport coefficients, obtained by the use of the slaving principle, giving us an overall (averaged) picture of the transport of the tracer in the Taylor vortex flow for long times.

6. Example: axial dispersion in the Davey–DiPrima–Stuart model for the Taylor vortex

We now apply these results to a specific model for the Taylor vortex due to Davy, DiPrima and Stuart [6]. This model is based on an asymptotic solution of the Navier–Stokes equations in the small gap limit. The derivation of the model is rather lengthy so we just present the form of the velocity field and refer the reader to the original paper of Davey, DiPrima and Stuart for more details.

The velocity field for the Taylor vortex flow is

$$u_x = \delta\Omega_0(-2\alpha(\delta T)^{1/2})A_c f_{20}(x)\cos(z) \tag{52}$$
$$u_\phi = \delta\Omega_0 d^{-1}(1 - \alpha x + A_c f_0(x)\cos(z)) \tag{53}$$
$$u_z = \delta\Omega_0(-2\alpha(\delta T)^{1/2})A_c f_{30}(x)\sin(z) \tag{54}$$

where A_c is a constant

$$\Omega_0 = \frac{\Omega_1 + \Omega_2}{2} \qquad \alpha = 2\left(\frac{1-\mu}{1+\mu}\right) \qquad \mu = \frac{\Omega_2}{\Omega_1} \tag{55}$$

Ω_1 is the angular velocity of the inner cylinder and Ω_2 is the angular velocity of the outer cylinder. The radii of the cylinders are R_1 and R_2 with $R_2 > R_1$ and

$$R_0 = \frac{R_1 + R_2}{2} \qquad d = R_2 - R_1 \qquad \delta = \frac{d}{R_0} \tag{56}$$

and the coordinates are defined as

$$R = R_0 + xd \qquad Z = zd \tag{57}$$

and (R, θ, Z) are cylindrical polar coordinates. T is the Taylor number. The functions f_0, f_{20} and f_{30} are functions of the radial coordinate x and are given by the solution of boundary value problems. For more details on the boundary value problems these functions will have to satisfy the reader is referred to [5] and [6].

In the small gap limit the advection diffusion equation simplifies to

$$\frac{\partial P}{\partial t} - \alpha_1 \frac{\partial}{\partial x}(uP) - \alpha_2 \frac{\partial}{\partial z}(wP) = \bar{D}_1 \frac{\partial^2 P}{\partial x^2} + \bar{D}_2 \frac{\partial^2 P}{\partial z^2} \tag{58}$$

where

$$\alpha_1 = \frac{\Omega_0 A_c}{\alpha\sqrt{Ta\delta}} \qquad \alpha_2 = \frac{2\lambda\Omega_0 A_c}{\alpha\sqrt{Ta\delta}}$$
$$\bar{D}_1 = \frac{D_{rr}}{\delta^2 R_0^2} \qquad \bar{D}_2 = \frac{D_{zz}\lambda^2}{\delta^2 R_0^2}$$
$$u = f_{20}(x)\cos(z) \qquad w = f_{30}\sin(z).$$

The boundary condition is the no-flux (Neumann) boundary condition at $x = \pm 0.5$. We notice that in the small gap limit the cylindrical geometry degenerates into plane geometry which simplifies immensely the solution of the eigenvalue problem in the x direction.

Using the notation of the previous sections we write $P = \sum_{m=0}^{\infty} A_m w_m(x)$ where now $w_m = \cos(m\pi x + (m\pi/2))$ and the eigenvalue spectrum is $\lambda_m = \bar{D}_1 m^2\pi^2$. The mode amplitudes satisfy the equations

$$\frac{\partial A_n}{\partial t} - \cos(z)\sum_m \alpha_1 \frac{\bar{\Delta}_{mn}}{\sigma_n} A_m - \sum_m \alpha_2 \frac{\Delta_{mn}}{\sigma_n}\frac{\partial}{\partial z}(\sin(z)A_m) - \bar{D}_1 n^2\pi^2 A_n + \bar{D}_2 \frac{\partial^2 A_n}{\partial z^2} \tag{59}$$

where

$$\Delta_{mn} = \int_{-0.5}^{0.5} f_{30} w_m w_n \,\mathrm{d}x$$
$$\bar{\Delta}_{mn} = -\int_{-0.5}^{0.5} f_{20} w_m \frac{\mathrm{d}w_n}{\mathrm{d}x}\,\mathrm{d}x.$$

The slaving relations in this case become

$$-2\cos(z)\sum_{m\neq 0}(\alpha_1\bar{\Delta}_{mn} + \alpha_2\Delta_{mn})A_m = -\bar{D}_1 n^2\pi^2 A_n + 2\alpha_2\Delta_{0n}\sin(z)\frac{\partial A_0}{\partial z} \tag{60}$$

and the diagonal approximation gives

$$A_n = 2\alpha\Delta_{0n}\sin(z)\frac{\partial A_0}{\partial z}(\bar{D}_1 n^2\pi^2 - 2(\alpha_1\bar{\Delta}_{nn} + \alpha_2\Delta_{nn})\cos(z))^{-1}. \tag{61}$$

Insertion of this result in the equation for the evolution of the neutral mode gives

$$\frac{\partial A_0}{\partial t} = \frac{\partial}{\partial z}(U_e A_0) + \frac{\partial}{\partial z}\left(D_e \frac{\partial A_0}{\partial z}\right) \tag{62}$$

where

$$U_e = 2\alpha_2\Delta_{00}\sin(z)$$
$$D_e = \bar{D}_2 + 2\alpha_2^2\sum_m \Delta_{m0}^2 \sin^2(z)(\bar{D}_1 m^2\pi^2 - a_m\cos(z))^{-1}$$
$$a_m = 2(\alpha_1\bar{\Delta}_{mm} + \alpha_2\Delta_{mm}).$$

To a first approximation (in the spirit of the previous section) the effective diffusion coefficient gives

$$D_e = \bar{D}_2 + 21.6\frac{\alpha_2^2}{\bar{D}_1}\sin^2(z) - 4.32\frac{\alpha_2^2\alpha_1}{\bar{D}_1^2}\cos(z)\sin^2(z) + 2.76\frac{\alpha_2^3}{\bar{D}_1^2}\cos(z)\sin^2(z) \tag{63}$$

whereas V_e vanishes.

We omit the third and fourth terms in this expansion and using the results summarized in equation (51) we can calculate the average diffusion coefficient which would be

$$D_a \simeq \bar{D}_2\left(1 + \frac{21.6\alpha_2^2}{\bar{D}_1\bar{D}_2}\right)^{1/2}. \tag{64}$$

A similar calculation can be performed very easily for the case of particles with inertia and the interesting result here would be that the average drift coefficient will no longer vanish.

7. Conclusions

In this paper we have proposed an alternative approach to the problem of Taylor dispersion using the slaving principle. The approach is easy to use even when the flow field is periodic in the direction along which dispersion is studied. Furthermore, it is mathematically transparent, hinging on physical intuition and thus enables us to consider more complicated physical situations.

In an attempt to validate and illustrate it, the approach was tested on the simple shear flow giving agreement with other works (e.g. [2]).

We then studied the problem of Taylor dispersion in the Taylor vortex, obtaining explicit expressions for the transport coefficients. Since in general the effective transport coefficients will be periodic functions of space, we have further proposed a way of deriving average transport coefficients for such systems which will model in a satisfactory manner the large scale evolution of the tracer probability distribution.

Finally, we have studied the effects of particle inertia, i.e. motion of a tracer with a density different from that of the fluid, on Taylor dispersion in the Taylor vortex, and given explicit expressions for the effective (local) transport coefficients of such tracers. As an example, we have presented results for the Taylor dispersion in the Davey–DiPrima–Stuart model for the Taylor vortex which is a perturbative solution of the full Navier–Stokes equations for the Taylor–Couette flow in the small gap limit.

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Appendix

In this appendix we present the results of the next iteration of our proposed iterative scheme from section 3. The second iteration for the amplitude of the modes gives

$$A_m^{(1)} = G(m,z)\left\{H(z,m,A_0) + \sum_{m\neq m',0} F(m,m',z)G(m',z)H(z,m',A_0)\right\}.$$

Expanding $G(m',z)$ about the dominant term which is λ_m, we find an approximate result using the second iteration which gives

$$A_m^{(1)} = G(m,z)\left\{H(z,m,A_0) + \sum_{m\neq m',0} F(m,m',z)\frac{1}{\lambda_{m'}} - \frac{F(m,m',z)}{\lambda_{m'}^2}\left(\frac{\bar{\Lambda}_{m'm'}}{\sigma_m'}Z_2'(z) - \frac{\Lambda_{m'm'}}{\sigma_m'}R_2(z)\right)H(z,m',A_0)\right\} + \mathrm{O}\left(\frac{1}{\lambda_{m'}^3}\right) \quad (65)$$

from which we see that the second iteration will contribute corrections of higher order in $1/D$ where D is the molecular diffusion.

We can now calculate the effect of the corrections due to the second iteration on the transport coefficients. Substituting (65) into (24) we find that the neutral mode satisfies the transport equation

$$\frac{\partial A_0}{\partial t} + \frac{\partial}{\partial z}(V_e(z)A_0) = \frac{\partial}{\partial z} D_e(z) \frac{\partial A_0}{\partial z}$$

where

$$\begin{aligned} V_e(z) = \Bigg(\sum_{m=1}^{\infty} \frac{\bar{\Lambda}_{m0}\Lambda_{0m}}{\sigma_0\sigma_m} G(m,z) + \sum_{m=1}^{\infty} \sum_{m\neq m',0} \frac{\bar{\Lambda}_{m0}\Lambda_{0m'}}{\sigma_0\sigma'_m} G(m,z)G(m',z)F(m,m',z) \Bigg) \\ \times R_2(z)Z_2(z) - \Bigg(\sum_{m=1}^{\infty} \frac{\bar{\Lambda}_{0m}^2}{\sigma_0\sigma_m} G(m,z) \\ + \sum_{m=1}^{\infty} \sum_{m\neq m',0} \frac{\bar{\Lambda}_{m0}\bar{\Lambda}_{0m'}}{\sigma_0\sigma_{m'}} G(m,z)G(m',z)F(m,m',z) \Bigg) Z_2(z)Z_2'(z) \end{aligned}$$

and

$$D_e(z) = D + \Bigg(\frac{\bar{\Lambda}_{0m}^2}{\sigma_0\sigma_m} G(m,z) + \sum_{m=1}^{\infty} \sum_{m\neq m',0} \frac{\bar{\Lambda}_{m0}\bar{\Lambda}_{0m'}}{\sigma_0\sigma_{m'}} G(m,z)G(m',z)F(m,m',z) \Bigg) Z_2(z)^2.$$

We now exploit the fact that λ_m are considerably larger than Λ_{mm} and $\bar{\Lambda}_{mm}$ to expand $G(m,z)$ in a Taylor series. This will give an approximate D_e of the form

$$\begin{aligned} D_e(z) = D + D_1 Z_2(z)^2 + D_2 Z_2(z)' Z_2^2(z) + D_3 R_2 Z_2(z)^2 + D_4 Z_2(z)' Z_2^2(z) \\ + D_5 R_2 Z_2(z)^2 + \mathrm{O}(D^{-3}) \end{aligned}$$

where D, D_1, D_2 and D_3 are the same coefficients as those obtained using the diagonal approximation (see section 3) and

$$D_4 = \sum_{m=1}^{\infty} \sum_{m\neq m',0} \frac{1}{\lambda_m\lambda_{m'}} \frac{\bar{\Lambda}_{m',m}}{\sigma_m} \qquad D_5 = -\sum_{m=1}^{\infty} \sum_{m\neq m',0} \frac{1}{\lambda_m\lambda_{m'}} \frac{\Lambda_{m',m}}{\sigma_m}.$$

These two coefficients are of order D^{-2} where D is the coefficient of molecular diffusion. The coefficients D_2 and D_3 are of the same order, while the coefficient D_1 is of order D^{-1}. It is seen that the second iteration will leave unchanged the results up to order D^{-1} and will only change the results to order D^{-2}. In a similar manner one can see that the next iteration of the scheme will introduce corrections of order D^{-3}.

A similar expansion can be performed for V_e.

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